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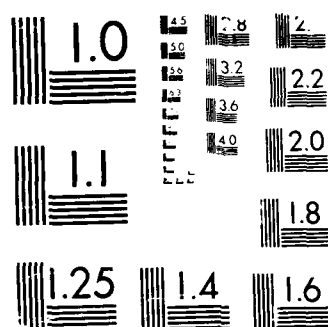
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ARISING IN THE THEORY OF M-ESTIMATES\*

Z. D. Bai, X. R. Chen,  
B. Q. Miao and Y. H. Wu

Center for Multivariate Analysis  
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Technical Report No. 87-46

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**Center for Multivariate Analysis**  
**University of Pittsburgh**



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ABSTRACT

This article, by obtaining the limit of probability that some equation arising in a case of M-estimate possesses at least one solution, establishes the fact that even in the simplest case, when the function  $\rho$  is not differentiable at least at one point, it is not legitimate to convert the minimization problem

$$\sum_{i=1}^n \rho(Y_i - x_i' \beta) = \min$$

defining the M-estimate to the solution of equations  $\sum_{i=1}^n \rho'(Y_i - x_i' \beta) x_i = 0$ .

*AMS 1980 Subject Classifications:* Primary 62J05.

*Key words and phrases:* M-estimate, linear model.

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20. (continued)

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$$\sum_{i=1}^n \rho(Y_i - x_i' \beta) = \min$$

defining the M-estimate to the solution of equations  $\sum_{i=1}^n \rho'(Y_i - x_i' \beta) x_i = 0$ .

## 1. INTRODUCTION

Consider a standard linear form

$$Y_i = x_i' \beta_0 + e_i, \quad i = 1, \dots, n, \dots \quad (1.1)$$

where  $\{x_i\}$  is a sequence of known  $p$ -vectors,  $p \geq 1$ ,  $\{e_i\}$  is a sequence of random errors. The M-estimate (Huber (1964, 1973))  $\hat{\beta}_n$  of  $\beta_0$  is defined as a solution of the minimization problem

$$\sum_{i=1}^n \rho(Y_i - x_i' \beta) = \min \quad (1.2)$$

$\rho$  is a chosen function on  $R^1$ . When  $\rho$  is convex and  $\rho'$  exists everywhere on  $R^1$ , (1.1) can be rewritten as the problem of solving the system of equations:

$$\sum_{i=1}^n \rho'(Y_i - x_i' \beta) x_i = 0. \quad (1.3)$$

Some authors obtain asymptotic properties of M-estimates via this approach. See Yohai and Maronna (1979), Maronna and Yohai (1981), Huber (1973), among others.

When  $\rho$  is not convex, but  $\rho'$  exists everywhere, the solution of (1.2) must still be a solution of (1.3). Therefore, if it can be shown that some asymptotic property is possessed by all solutions of (1.3), then this property is also possessed by the solution of (1.2). Only in such circumstances one has the right to assert that the problem (1.2) can be converted to (1.3).

However, in many cases of practical significance,  $\rho$  is not differentiable somewhere. A famous example is  $\rho(u) = |u|$ , leading to the much-studied Minimum  $L_1$ -Norm estimate. In such cases one may still write down

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the equation (1.3) formally, ignoring for the moment the fact that  $\rho'(u)$  is not defined for some  $u$ . To justify such an approach, one has to make sure the following two points:

- 1) That the probability of the event {(1.3) has a solution} tends to one as  $n \rightarrow \infty$ .
- 2) That the probability of the event {The solution mentioned in 1) is a solution of (1.2)} tends to one as  $n \rightarrow \infty$ .

Since it is difficult to study the M-estimate directly resorting to the minimization (1.2), it makes good sense to give a closer look into the problem: Whether or not it is possible to validate this approach in a reasonably general framework? The present article solves this problem in the negative.

This work is stimulated by a paper of Dodge and Jureckova (1987), who studied the M-estimate defined by the function  $\rho(u) = \delta|u| + (1 - \delta)u^2$  ( $0 \leq \delta \leq 1$ ). This function is not differentiable at  $u = 0$ , unless  $\delta = 0$ . But they used (1.3) to replace the minimization problem (1.2) without clarifying the two points mentioned above. We shall show that even in the simplest special case of estimating a location parameter, their approach is invalid, to say nothing about the general case. Our result shows that to obtain rigorous results of M-estimate when  $\rho$  is not everywhere differentiable, one must pay due attention to the original minimization problem (1.2). See, for example, Chen and Wu (1987).

## 2. FORMULATION OF THE THEOREM

Consider a special case of (1.1) in which  $\beta_0$  is one-dimensional,  $x_1 = x_2 = \dots = 1$ . The problem becomes one of estimating the location parameter. Since  $\rho(u) = \delta|u| + (1-\delta)u^2$ , on writing  $\beta_0 = a$ , the equation (1.3) assumes the form

$$\delta \sum_{i=1}^n \operatorname{sgn}(Y_i - a) + 2(1-\delta) \sum_{i=1}^n (Y_i - a) = 0 \quad (2.1)$$

where  $\operatorname{sgn}(0) = 0$ ,  $\operatorname{sgn}(u) = u/|u|$  for  $u \neq 0$ . Define

$$E_n = \text{the event } \{(2.1) \text{ has at least one solution}\}, \quad p_n = P(E_n). \quad (2.2)$$

We have the following theorem:

**THEOREM.** Suppose that  $Y_1, Y_2, \dots$  are independent and identically distributed (i.i.d.) random variables with a finite variance and a common density function  $f$ . Denote by  $c$  the unique solution of the minimization problem for  $0 < \delta < 1$ :

$$Q(a) \equiv \delta E|Y_i - a| + (1-\delta)E(Y_i - a)^2: = \min. \quad (2.3)$$

Suppose that  $f$  is continuous and positive at  $c$ . Then

$$\lim_{n \rightarrow \infty} p_n = (1-\delta)/(1-\delta+\delta f(c)) \quad (2.4)$$

which is less than one for  $\delta \in (0,1)$ , and can be arbitrarily small when  $f(c)$  is large, or  $\delta$  is close to 1.

## 3. A CRUDE RESULT

We shall give the proof of the Theorem in Section 4. Since the proof is quite devious, for readers who are not interested in the specific value of  $\lim p_n$ , we offer in this section a simple and elementary proof of a less-accurate result, from which, nevertheless, follows the invalidity of the approach of Dodge and Jureckova (1987): Suppose that  $Y_1, Y_2, \dots$  are i.i.d. variables with a common rectangular distribution  $R(0,1)$  and  $\delta = 2/3$ , then we have

$$\liminf_{n \rightarrow \infty} (1 - p_n) > 0. \quad (3.1)$$

For a proof, denote by  $\xi_{n1} < \xi_{n2} < \dots < \xi_{nn}$  the order statistics of  $Y_1, \dots, Y_n$ , and hitherto we shall abbreviate  $\xi_{ni}$  to  $\xi_i$ . Write

$$W_n = \sum_{i=1}^{[n/2 + \sqrt{n}]} \xi_i / [n/2 + \sqrt{n}], \quad \eta_n = \sum_{i=[n/2 + \sqrt{n}] + 1}^n \xi_i,$$

where  $[b]$  denotes the integer part of  $b$ . Remembering  $\delta = 2/3$ , rewrite (2.1) as

$$\sum_{i=1}^n \operatorname{sgn}(\xi_i - a) + \sum_{i=1}^n (\xi_i - a) = 0. \quad (3.2)$$

Write  $L_{n1}$  and  $L_{n2}$  for the intervals  $[1/2 - 2/\sqrt{n}, 1/2]$  and  $[1/2, 1/2 + 2/\sqrt{n}]$  respectively, and define the events

$$A_1 \equiv A_{1n} = \{\xi_{[n/2 - \sqrt{n}]} \in L_{n1}\},$$

$$A_2 \equiv A_{2n} = \{\xi_{[n/2 + \sqrt{n}]} \in L_{n2}\},$$

$$B \equiv B_n = \{W_n \in [\xi_{[n/2 + \sqrt{n}]} / 2 - 1/\sqrt{n}, \xi_{[n/2 + \sqrt{n}]} / 2 + 1/\sqrt{n}]\}.$$

Note that when  $A_2 \cap B$  occurs,  $W_n \in L_{n3} \equiv [1/4 - 1/\sqrt{n}, 1/4 + 2/\sqrt{n}]$ .

Denoting by  $\Phi$  the distribution function of  $N(0,1)$  and noticing the fact

that the conditional distribution of  $[n/2 + \sqrt{n}]W_n$  given  $\xi_{[n/2 + \sqrt{n}]}$  is the same as  $\xi_{[n/2 + \sqrt{n}]}$  plus the sum of  $[n/2 + \sqrt{n}] - 1$  i.i.d. variables with a common distribution  $R(0, \xi_{[n/2 + \sqrt{n}]})$ , it is easy to see that

$$\begin{aligned} \liminf_{n \rightarrow \infty} P(A_1 \cap A_2 \cap B) &\geq \lim_{n \rightarrow \infty} (P(A_1) + P(A_2) + P(B) - 2) \\ &= 3[\phi(2) - \phi(-2)] - 2 \equiv q > 0. \end{aligned} \quad (3.3)$$

Now we proceed to show that there exists constant  $r > 0$  such that when  $n$  is sufficiently large

$$P(E_n | \xi_{[n/2 + \sqrt{n}]} = u_n, \xi_{[n/2 - \sqrt{n}]} = v_n, W_n = w_n) \geq r \quad (3.4)$$

holds uniformly for  $u_n \in L_{n2}$ ,  $v_n \in L_{n1}$  and  $w_n \in L_{n3}$ . For this purpose write  $c_n = n - [n/2 + \sqrt{n}]$ . Under the above given conditions, the conditional distribution of  $\eta_n$  is the same as the sum of  $c_n$  i.i.d. variables with a common distribution  $R(u_n, 1)$ . Since  $1 - u_n \geq 1/3$ , when  $u_n \in L_{n2}$ , by elementary calculus, or employing local limit theorem (see Petrov (1975)), it can easily be shown that the conditional density  $f(t, u_n)$  of

$$Q_n \equiv 2\sqrt{3}(1 - u_n)^{-1} \sqrt{c_n}^{-1} (\eta_n - (1 + u_n)c_n/2) \quad (3.5)$$

tends uniformly to  $\sqrt{2\pi}^{-1} \exp(-t^2/2)$  for  $u_n \in L_{n2}$  and  $t$  bounded.

Since  $[n/2 + \sqrt{n}]W_n + \eta_n = \xi_1 + \dots + \xi_n$ , an inspection of the equation (3.2) reveals that when  $W_n$  held fixed, each real  $a$  corresponds one and only one value  $\eta_n(a)$  such that when  $\eta_n$  assumes the value  $\eta_n(a)$ , (3.2) will have solution  $a$ . Evidently  $\eta_n(a)$  increases linearly with  $a$  as long as  $a$  stays within  $(\xi_i, \xi_{i+1})$  for some  $i$ , and  $\eta_n(a)$  has a jump of magnitude 2 at  $a = \xi_i$ . As  $a$  runs from  $\xi_{[n/2 - \sqrt{n}]}$  to  $\xi_{[n/2 + \sqrt{n}]}$ ,  $a$  passes at least  $2\sqrt{n} - 3$  such  $\xi_i$ 's. Hence there exist at least  $2\sqrt{n} - 3$  intervals, each with length 2,

and any two of them are disjoint, such that if  $\eta_n$  falls into one of these intervals but not equal to the midpoint of this interval, (3.2) has no solution. Denote by  $D_n$  the union of these intervals. It is easy to see that

$$D_n \subset [3n/8 - 21\sqrt{n}/4 - 3, 3n/8 + 17\sqrt{n}/4] \quad (3.6)$$

when  $u_n \in L_{n2}$ ,  $v_n \in L_{n1}$  and  $w_n \in L_{n3}$ . In fact, from equation (3.2) it is seen that when  $u_n, v_n, w_n$  satisfy these conditions, each point in  $D_n$  must not be less than

$$\begin{aligned} & \left( na - (n/2 + \sqrt{n})w_n - \sum_{i=1}^n \operatorname{sgn}(\xi_i - \xi_{[n/2 - \sqrt{n}]}) \right) \Big|_{a=1/2 - 2/\sqrt{n}, w_n=1/4 + 2/\sqrt{n}} \\ &= n(1/2 - 2/\sqrt{n}) - (n/2 + \sqrt{n})(1/4 + 2/\sqrt{n}) - (n - 2[n/2 - \sqrt{n}] + 1) \\ &\geq 3n/8 - 21\sqrt{n}/4 - 3. \end{aligned}$$

The other end is similar. From (3.5), (3.6) and the fact that  $u_n \in L_{n2}$ , it is easy to see that  $\eta_n \in D_n$  entails  $|Q_n| \leq 50 + o(1)$ . Denote by  $|B|$  the Lebesgue measure of a set  $B$ . We have seen that  $|D_n| \geq 4\sqrt{n} - 6$ . Hence by (3.5) and  $u_n \in L_{n2}$ , we have  $|\{Q_n: \eta_n \in D_n\}| \geq 38 + o(1)$ . These facts, and the fact that  $f(u_n, t)$  converges uniformly to  $\sqrt{2\pi}^{-1} \exp(-t^2/2)$ , enable us to infer that when  $n$  is sufficiently large,

$$\text{The left-hand side of (3.4)} \geq 2^{-1} \sqrt{2\pi}^{-1} \int_{31 \leq |t| \leq 50} e^{-t^2/2} dt \equiv r$$

uniformly for  $u_n \in L_{n2}$ ,  $v_n \in L_{n1}$ ,  $w_n \in L_{n3}$ . This proves (3.4). Now (3.3) and (3.4) together give

$$1 - p_n \geq rq/2 > 0$$

for  $n$  large, and (3.1) is proved.

## 4. PROOF OF THE THEOREM

The following lemma will be needed in the proof.

LEMMA 1. Suppose that we have a triangular array of random variables  $\{X_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ , such that  $\lim_{n \rightarrow \infty} m_n = \infty$  and for each  $n$ ,  $X_{n1}, \dots, X_{nm_n}$  are independent. Suppose that  $EX_{ni} = 0$ ,  $0 < \sigma_{ni}^2 = \text{var}(X_{ni}) < \infty$ ,  $1 \leq i \leq m_n$ ,  $n \geq 1$ . Write

$$\sigma_n^2 = \sum_{i=1}^{m_n} \sigma_{ni}^2, \quad X_n = \sum_{i=1}^{m_n} X_{ni}, \quad F_n(x) = P(X_n \leq \sigma_n x)$$

and assume that the following conditions are satisfied:

- a. There exist constants  $\Delta_1 > 0$ ,  $\Delta_2 > 0$ , such that

$$\Delta_1 \leq \sigma_{ni}^2 \leq \Delta_2, \quad \text{for } 1 \leq i \leq m_n, \quad n \geq 1.$$

- b. There exist positive constants  $c_1, c_2, \dots$  tending to 0, such that

$$\sup_{|t| \leq 1/m} \sup_{n \geq 1} \sup_{1 \leq i \leq m_n} |f_{ni}''(t) + \sigma_{ni}^2| \leq c_m, \quad m \geq 1,$$

where  $f_{ni}$  is the characteristic function of  $X_{ni}$ .

- c. There exists a positive function  $g = g(a, b)$  defined on the set  $\{(a, b): 0 < a < b\}$ , such that

$$\sup_{a \leq |t| \leq b} |f_{ni}(t)| \leq \exp(-g(a, b)), \quad \text{for } 1 \leq i \leq m_n, \quad n \geq 1.$$

Then we have

$$\sup_{|x| < \infty} |(F_n(x+h) - F_n(x)) - (\Phi(x+h) - \Phi(x))| = o(h + \sqrt{n}^{-1}).$$

Here  $o(h + \sqrt{n}^{-1})$  depends only upon  $\Delta_1, \Delta_2, \{c_m\}$ , and the function  $g$ .

Stone (1965) proved this lemma in the special case that

$$m_n = n, \quad X_{ni} = X_i, \quad X_1, X_2, \dots \text{ are i.i.d. variables.}$$

His method of proof can be adopted here with some minor modifications, so

the details are omitted.

Now turn to the proof of the theorem. Denote by  $\mu$ ,  $m$  and  $\sigma^2$  the expectation, median and variance of  $Y_1$  respectively. Without loss of generality, assume that  $c = 0$  (see (2.4)). Since  $c$  must lie between  $\mu$  and  $m$ , without loss of generality assume that  $\mu \leq 0 \leq m$ . Denote by  $F$  the distribution function of  $Y_1$ , and  $q = F(0)$ , we have  $0 < q \leq 1/2$ . Since 0 is the minimization point of  $Q(a)$ , we easily verify that

$$\mu = -\lambda(1-2q) \quad \left( \lambda = \delta / (2(1-\delta)) \right). \quad (4.1)$$

Denote, as in Section 3, by  $\xi_1 < \dots < \xi_n$  the order statistics of  $Y_1, \dots, Y_n$ . Evidently, if (2.1) has a solution  $\hat{a}_n$ , there must be some  $j$  such that

$$\xi_j \leq \hat{a}_n < \xi_{j+1}. \quad (4.2)$$

Since  $F$  is a continuous distribution, it is easy to verify that  $P(\hat{a}_n = \xi_j \text{ for some } j) = 0$ , and (4.2) may be replaced by

$$\xi_j < \hat{a}_n < \xi_{j+1}. \quad (4.3)$$

Considering the equation (2.1), (4.3) is equivalent to

$$\xi_j < \bar{Y}_n + \lambda(n-2j)/n < \xi_{j+1}, \quad \bar{Y}_n = \sum_{i=1}^n Y_i/n. \quad (4.4)$$

Writing  $p_{nj} = P(\xi_j < \bar{Y}_n + \lambda(n-2j)/n < \xi_{j+1})$ ,  $j = 0, 1, \dots, n$ , with the convention that  $\xi_0 = -\infty$ ,  $\xi_{n+1} = \infty$ , we have

$$p_n = p_{n0} + p_{n1} + \dots + p_{nn}. \quad (4.5)$$

Now fix  $M > 0$ , and write  $j_1 = j_{1n} = [qn - M\sqrt{n}]$ ,  $j_2 = j_{2n} = [qn + M\sqrt{n}]$  ( $q = F(0)$ ). We verify easily that: If  $\xi_{j_2} \geq 0$ ,  $|\bar{Y}_n - \mu| \leq M/\sqrt{n}$ , then

(4.4) does not hold for  $j > j_2$ . Similarly, if  $\xi_{j_1} \leq 0$ ,  $|\bar{Y}_n - \mu| \leq \lambda M/\sqrt{n}$ , (4.4) does not hold for  $j < j_1$ . Hence

$$\sum_{j=1}^{j_1-1} p_{nj} + \sum_{j=j_2+1}^n p_{nj} \leq P(|\bar{Y}_n - \mu| \geq \lambda M/\sqrt{n}) + P(\xi_{j_1} \geq 0) + P(\xi_{j_2} \leq 0). \quad (4.6)$$

From (4.6), it follows easily that

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \left( \sum_{j=1}^{j_1-1} p_{nj} + \sum_{j=j_2+1}^n p_{nj} \right) = 0. \quad (4.7)$$

For any integer  $j \in [j_1, j_2]$  and  $x < y$ , define

$$p_{nj}(x, y) = P(x < \bar{Y}_n + \lambda(n-2j)/n < y | \xi_j = x, \xi_{j+1} = y).$$

Since the conditional distribution of  $\bar{Y}_n$  given  $\xi_j = x$ ,  $\xi_{j+1} = y$  is the same as the distribution of  $(x + y + \zeta_1 + \dots + \zeta_{j-1} + \eta_1 + \dots + \eta_{n-j-1})/n$ , where  $\zeta_1, \dots, \zeta_{j-1}, \eta_1, \dots, \eta_{n-j-1}$  are independent,  $\zeta_1, \dots, \zeta_{j-1}$  are i.i.d. with a common density  $I_{(-\infty, x)}(\cdot)f(\cdot)/F(x)$  with expectation  $\mu_{1x}$  and variance  $\sigma_{1x}^2$ ,  $\eta_1, \dots, \eta_{n-j-1}$  are i.i.d. with a common density  $I_{(y, \infty)}(\cdot)f(\cdot)/[1 - F(y)]$  with expectation  $\mu_{2y}$  and variance  $\sigma_{2y}^2$ , on writing

$$R_n = \left( \sum_{i=1}^{j-1} (\zeta_i - \mu_{1x}) + \sum_{i=1}^{n-j-1} (\eta_i - \mu_{2y}) \right) / S_n, \quad S_n^2 = (j-1)\sigma_{1x}^2 + (n-j-1)\sigma_{2y}^2,$$

we have

$$R_n \xrightarrow{\mathcal{L}} N(0, 1), \quad \text{as } n \rightarrow \infty. \quad (4.8)$$

Now apply Lemma 1, putting  $\{X_{n1}, \dots, X_{nm_n}\} = \{\tilde{\zeta}_1, \dots, \tilde{\zeta}_{j-1}, \tilde{\eta}_1, \dots, \tilde{\eta}_{n-j-1}\}$ , where  $\tilde{\zeta}_i = \zeta_i - \mu_{1x}$ ,  $\tilde{\eta}_i = \eta_i - \mu_{2y}$ . Since  $Y_1$  possesses a density function  $f$ ,  $f(0) > 0$  and  $f$  is continuous at  $x = 0$ , it can easily be verified that for  $j \in [j_1, j_2]$ ,  $|x| < n^{-\alpha}$ ,  $|y| < n^{-\alpha}$ , all conditions of Lemma 1 are satisfied, and furthermore, the quantities  $\Delta_1, \Delta_2$ , sequence  $\{c_m\}$  and



function  $g$  mentioned in that lemma can be chosen in such a way that they are independent of  $j, x, y$ , as long as  $j \in [j_1, j_2]$ ,  $|x| < n^{-\alpha}$  and  $|y| < n^{-\alpha}$ . Therefore we obtain

$$P_{nj}(\xi_j, \xi_{j+1}) = (\phi(b_j) - \phi(a_j))(1 + o(1)) + o(n^{-1/2}). \quad (4.9)$$

Here  $o(1)$  and  $o(n^{-1/2})$  are uniform for  $j \in [j_1, j_2]$ , and

$$a_j = \{\xi_j - (\xi_j + \xi_{j+1})/n - [(j-1)\mu_{1\xi_j} + (n-j-1)\mu_{2\xi_{j+1}}]/n - \lambda(n-2j)/n\}nS_n^{-1}, \quad (4.10)$$

$$b_j = a_j + (\xi_{j+1} - \xi_j)nS_n^{-1}. \quad (4.11)$$

Since  $f(0) > 0$  and  $f$  is continuous at 0, it follows easily by an inequality by Bennett (1962) that for any  $\alpha \in (0, 1/2)$ :

$$P(|\xi_j| \geq n^{-\alpha}) = o(n^{-1}) \quad (4.12)$$

uniformly for  $j \in [j_1, j_2]$ , as  $n \rightarrow \infty$ . On the other hand, denoting by  $F_j$  the distribution function of  $\xi_j$ , we have

$$\begin{aligned} & P(\xi_{j+1} - \xi_j > n^{-2\alpha} \mid |\xi_j| < n^{-\alpha}) \\ &= (P(|\xi_j| < n^{-\alpha}))^{-1} \int_{|x| \leq n^{-\alpha}} [1 - f(0)(1 + o(1))n^{-2\alpha}/(1 - q)]^{n-j} dF_j(x) \\ &\leq \exp\{-2^{-1}f(0)(1 + o(1))n^{1-2\alpha}\} = o(n^{-1}) \end{aligned} \quad (4.13)$$

uniformly for  $j = 1, \dots, n$ , as  $n \rightarrow \infty$ .

Now we proceed to make estimates on  $\mu_{1x}$ ,  $\mu_{2y}$  and  $S_n^2$ . For this purpose introduce the following notations:

$$u_1 = E(Y_1 I_{(Y_1 \leq 0)}),$$

$$u_2 = E(Y_1 I_{(Y_1 \geq 0)}).$$

We have

$$\begin{aligned}
 S_n^2 &= (j-1)\sigma_{1x}^2 + (n-j-1)\sigma_{2y}^2 \\
 &= qn(1+o(1))\left(q^{-1}\int_{u\leq 0} u^2 dF(u) - q^{-2}u_1^2\right)(1+o(1)) \\
 &\quad + (1-q)n(1+o(1))\left((1-q)^{-1}\int_{u\geq 0} u^2 dF(u) - (1-q)^{-2}u_2^2\right)(1+o(1)) \\
 &= n(EY_1^2 - q^{-1}u_1^2 - (1-q)^{-1}u_2^2)(1+o(1)) \equiv n\sigma_0^2(1+o(1))
 \end{aligned} \tag{4.14}$$

where

$$\sigma_0^2 = \sigma^2 - q^{-1}(1-q)^{-1}((1-q)u_1 - qu_2)^2 > 0.$$

Choose  $\alpha \in (1/4, 1/2)$ . When  $|x| < n^{-\alpha}$ , we have

$$\begin{aligned}
 \mu_{1x} &= \int_{-\infty}^x u dF(u)/F(x) = (u_1 + O(n^{-2\alpha}))(q + \int_0^x dF)^{-1} \\
 &= q^{-1}(u_1 - q^{-1}u_1 \int_0^x dF + O(n^{-2\alpha})).
 \end{aligned} \tag{4.15}$$

Similarly, when  $|x| < n^{-\alpha}$  and  $|y-x| < n^{-2\alpha}$ , we have

$$\mu_{2y} = (1-q)^{-1}\left(u_2 + (1-q)^{-1}u_2 \int_0^x dF + O(n^{-2\alpha})\right). \tag{4.16}$$

Writing  $j = [qn] + t_j$ , for  $j \in [j_1, j_2]$ , and  $\omega = [(1-q)u_1 - qu_2]/(q(1-q))$ , from (4.10), (4.12)-(4.16), after some simplifications we get

$$a_j = \sigma_0^{-1}\sqrt{n}\{(1+\omega f(0))\xi_j + (2\lambda - \omega)t_j/n\}(1+o(1)). \tag{4.17}$$

Here  $o(1) \rightarrow 0$  uniformly for  $j \in [j_1, j_2]$ . For  $b_j$  we get exactly the same expression as (4.17) — of course, with a different  $o(1)$ . From (4.10), (4.11) and (4.14), we have  $b_j - a_j = n(\xi_{j+1} - \xi_j)/S_n = \sigma_0^{-1}\sqrt{n}(\xi_{j+1} - \xi_j)(1+o(1))$ . Hence, from (4.9), we obtain

$$p_{nj}(\xi_j, \xi_{j+1}) = \sqrt{n} \sigma_0^{-1} (\xi_{j+1} - \xi_j) \phi \left( \sqrt{n} \sigma_0^{-1} \left( (1 + \omega f(0)) \xi_j + (2\lambda - \omega) t_j/n \right) (1 + o(1)) \right) + o(n^{-1/2}), \quad (4.18)$$

where  $o(n^{-1/2})$  is uniform for  $j \in [j_1, j_2]$ , and  $\phi(u) = \sqrt{2\pi}^{-1} \exp(-u^2/2)$ .

Now we show that

$$E(\xi_{j+1} - \xi_j | \xi_j = a) = (nf(0))^{-1} (1 + o(1)) \quad (4.19)$$

uniformly for  $j \in [j_1, j_2]$  and  $|a| < n^{-\alpha}$ , as  $n \rightarrow \infty$ . In fact,

$$E(\xi_{j+1} - \xi_j | \xi_j = a) = \left( \int_a^{a+n^{-2/3}} + \int_{a+n^{-2/3}}^{\infty} \right) \left( \frac{1 - F(x)}{1 - F(a)} \right)^{n-j} dx \equiv J_{1n} + J_{2n}. \quad (4.20)$$

Since  $|a| < n^{-\alpha}$  and  $n - j = n(1 - q)(1 + o(1))$  uniformly for  $j \in [j_1, j_2]$ , we have

$$\begin{aligned} \left[ (1 - F(x)) / (1 - F(a)) \right]^{n-j} &= \left[ 1 - (1 - q)^{-1} f(0)(x - a)(1 + o(1)) \right]^{n-j} \\ &= \exp\{-nf(0)(x - a)(1 + o(1))\} \end{aligned} \quad (4.21)$$

uniformly for  $x \in [a, a + n^{-2/3}]$  and  $j \in [j_1, j_2]$ . If  $x > a + n^{-2/3}$ , the same argument gives

$$\begin{aligned} \left[ (1 - F(x)) / (1 - F(a)) \right]^{n-j-1} &\leq \left[ (1 - F(a + n^{-2/3})) / (1 - F(a)) \right]^{n-j-1} \\ &= \exp(-n^{1/3} f(0)(1 + o(1))) = o(n^{-1}) \end{aligned} \quad (4.22)$$

uniformly for  $x > a + n^{-2/3}$  and  $j \in [j_1, j_2]$ . From (4.20)-(4.22) and notice the fact that  $\int_1^{\infty} [1 - F(x)] dx \rightarrow \infty$  (because  $E|Y_1| < \infty$ ), (4.19) follows. From (4.19) we obtain

$$p_{nj} = E p_{nj}(\xi_j, \xi_{j+1}) = (\sigma_0 f(0) \sqrt{n})^{-1} E \phi \left( \sqrt{n} \sigma_0^{-1} \left( (1 + \omega f(0)) \xi_j + (2\lambda - \omega) t_j/n \right) (1 + o(1)) \right) + o(n^{-1/2}). \quad (4.23)$$

Write  $\sigma_1^2 = \text{var}(I_{(Y_1 \leq 0)}) = q - q^2 = q(1 - q)$ . Fix  $x$  and consider

$$\begin{aligned}
 & P\left(f(0)\sigma_1^{-1}\sqrt{n}\left(\xi_j - t_j/(nf(0))\right) < x\right) \\
 &= P\left(\sum_{i=1}^n \left\{ I\left(Y_i \leq (t_j + \sigma_1 x \sqrt{n})/(nf(0))\right) - F\left((t_j + \sigma_1 x \sqrt{n})/(nf(0))\right) \right\}\right) \\
 &\geq j - nF\left((t_j + \sigma_1 x \sqrt{n})/(nf(0))\right) \\
 &= P\left(\sum_{i=1}^n \left\{ I\left(Y_i \leq (t_j + \sigma_1 x \sqrt{n})/(nf(0))\right) - F\left((t_j + \sigma_1 x \sqrt{n})/(nf(0))\right) \right\} / (\sqrt{n}\sigma_1)\right) \\
 &\geq -x(1 + o(1)) + o(t_j n^{-1/2}). \tag{4.24}
 \end{aligned}$$

Since  $t_j/\sqrt{n} = O(1)$  uniformly for  $j \in [j_1, j_2]$ , the last expression of (4.24) tends to  $\phi(x)$  uniformly for  $j \in [j_1, j_2]$ , as  $n \rightarrow \infty$ . Denote by  $Z$  a random variable with distribution  $\phi$ . From (4.23), (4.24), and Helly's theorem, we get

$$\begin{aligned}
 p_{nj} &= (\sigma_0 f(0) \sqrt{n})^{-1} E\phi\left(\left[\sigma_1(1 + \omega f(0))\sigma_0^{-1}f^{-1}(0)Z + (1 + 2\lambda f(0)t_j)/(f(0)\sqrt{n}\sigma_0)\right]\right) \\
 &\quad + o(n^{-1/2}), \tag{4.25}
 \end{aligned}$$

where  $o(n^{-1/2})$  is uniform for  $j \in [j_1, j_2]$ . From (4.25) we obtain

$$\begin{aligned}
 \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=j_1}^{j_2} p_{nj} &= (\sigma_0 f(0))^{-1} \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=-[M\sqrt{n}]}^{[M\sqrt{n}]} E\phi\left(\frac{\sigma_1(1 + \omega f(0))}{\sigma_0 f(0)} Z + \frac{1 + 2\lambda f(0)}{\sigma_0 f(0)} \frac{i}{\sqrt{n}}\right) \frac{1}{\sqrt{n}} \\
 &= (\sigma_0 f(0))^{-1} \int_{-\infty}^{\infty} E\phi\left(\frac{\sigma_1(1 + \omega f(0))}{\sigma_0 f(0)} Z + \frac{(1 + 2\lambda f(0))}{\sigma_0 f(0)} u\right) du \\
 &= (2\pi\sigma_0 f(0))^{-1} \iint_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{\sigma_1(1 + \omega f(0))}{\sigma_0 f(0)} Z + \frac{1 + 2\lambda f(0)}{\sigma_0 f(0)} u\right)^2 - \frac{Z^2}{2}\right\} dZ du \\
 &= (1 + 2\lambda f(0))^{-1} + (1 - \delta)/(1 - \delta + \delta f(0)). \tag{4.26}
 \end{aligned}$$

Finally, from (4.7) and (4.26) we get (2.4). The theorem is proved.

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